FULLY CONSERVATIVE SPH-SYSTEMS IN GAS DYNAMICS AND A SCHEME FOR NUMERICALLY INTEGRATING THEM

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Fully conservative systems are identified among a set of ordinary differential equation systems of the smoothed particles hydrodynamics (SPH) method describing two-dimensional axially symmetric flows. A fully conservative SPH system has been constructed with the variational method and is a "duplicate" of the known Brookshaw system for equations in cylindrical coordinates. A modified *predictor-corrector* method for the numerical integration of SPH systems, which is based on the fully conservative central difference scheme of Lagrangian fluid dynamics, is proposed and it is demonstrated how the method can be used to simulate 2D axially symmetric gas flows.

Keywords: SPH method, axially symmetric flows, least action principle, Lagrangian gas dynamics, central difference schemes, modified *predictor—corrector* method, blast problem.

Introduction

We discuss the use of the Smoothed-Particle Hydrodynamics (SPH) method [1, 2] in two-dimensional planar and axially symmetric gas dynamics simulations, consider several systems of ordinary differential equations of the method in terms of their full conservativeness, and propose a fully conservative scheme for numerical integration of the SPH systems.

The concept of full conservativeness was introduced in Ref. [3] for gas-dynamic difference schemes. A difference scheme is called fully conservative, if, first, it satisfies the laws of conservation of mass and total energy, and the law of variation of momentum, and, second, provides proper balance between internal and kinetic energy. One of the necessary criteria of full conservativeness is the requirement that one should be able to derive a conservative difference equation of total energy from the difference equations of motion and internal energy, and from the equations of motion and entropy. That such a transformation is possible was demonstrated by Harlow in the difference scheme of the Particle-in-Cell method [4]. Ref. [4] also draws attention to the necessity of strict compliance with the laws of conservation when using non-divergent difference equations.

Of course, the definition of full conservativeness is also applicable to other gas dynamics discretizations, in particular, SPH. The SPH method consists of two discretization steps: in space and in time. The domain of interest is first represented as a set of macroparticles possessing mass, velocity, density, internal energy, and specified material parameters, and gas-dynamic SPH equations are defined as a system of ordinary differential equations. This system is then integrated over time. Note that if the SPH system is fully conservative, then it is natural to require that its integration scheme must preserve this property.

Different schemes of numerical solution can be used, for example, the Euler method, the improved Euler method, the Euler—Cauchy method, the Euler—Cauchy method with iterations, and other approaches. In this paper, we propose a new SPH scheme of numerical integration.

Note that the analysis of SPH schemes is a complex task in the general case, so such schemes are usually analyzed on less sophisticated models, for example, on difference equations. Based on the way of definition of mesh quantities, gas-dynamic difference schemes can be divided into node-centered and cell-centered schemes. In the node-centered schemes, velocities are defined at cell nodes (edges or faces), while the other quantities are defined at cell centers. An example of such a scheme is the scheme presented in [5]. However, the node-centered schemes cannot be used for modeling, because unknown quantities in the

SPH method are assigned to the center of mass of the macro-particle. So a suitable model must be sought among cell-centered difference schemes. In such schemes, all the quantities are assigned to cell centers. An example of such a scheme is Harlow's fully-conservative scheme [4]; unfortunately, it is unstable.

In this work, based on the analysis of the SPH method, we propose a conditionally stable cell-centered difference scheme possessing the property of full conservativeness. Based on it, we construct a fully conservative numerical integration method for the SPH equations. In general, the method is applicable in arbitrary orthogonal coordinates. In the paper, it is presented in cylindrical coordinates for an axially symmetric gas flow.

Note that the SPH representation of gas-dynamic equations, generally speaking, is non-unique, primarily, due to the formalism of the SPH method admitting different representations of basic gas-dynamic equations. In addition, in cylindrical coordinates, density interpolation also involves non-uniqueness. In the domain of the interpolation kernel, density can be either defined as a space average, or found based on the average of "planar" densities. Each of these approaches yields its own system of gas-dynamic SPH equations. Among diverse gas-dynamic SPH systems, preference is given to fully conservative schemes resulting from the Hamilton—Ostrogradksy variational principle of least action.

The paper consists of four sections. The first section states the problem and philosophy behind the SPH method. The second section presents several systems of gas-dynamic SPH equations in cylindrical coordinates, gives a definition of fully-conservative SPH schemes, provides examples of a non-conservative system and fully or simply conservative systems, and considers a new fully conservative SPH system constructed by the variational method. In the third section, a cell-centered difference scheme for one-dimensional gas dynamics is considered, which models the SPH method with a small number of particles, and a fully conservative scheme for numerical integration of the SPH equations is developed on its basis. The final section presents a numerical example.

1. Problem statement. Philosophy of the SPH method

Consider a flow of a non-heat-conducting ideal gas described by the following partial differential equations:

$$\frac{d\vec{u}}{dt} + \frac{1}{\rho}\nabla P = 0; \tag{1}$$

$$\frac{d\rho}{dt} + \rho\nabla\cdot\vec{u} = 0; \tag{2}$$

$$\frac{dE}{dt} + \frac{P}{\rho}\nabla\cdot\vec{u} = 0; \tag{3}$$

$$\frac{d\vec{r}}{dt} = \vec{u} \tag{4}$$

with the equation of state

$$P = P(\rho, S); \quad E = E(\rho, S), \tag{5}$$

where t is the time; $\vec{r}$ is the radius vector of the point; $\vec{u}$ is the velocity; ρ is the density, E is the specific internal energy, P is the pressure, and S is the entropy.

In the two-dimensional case,

$$\vec{r} = x\vec{l_x} + y\vec{l_y}; \qquad \vec{u} = u\vec{l_x} + v\vec{l_y}; \qquad \nabla\cdot\vec{u} = \frac{1}{y^\nu}\left(\frac{\partial}{\partial x}(y^\nu u) + \frac{\partial}{\partial y}(y^\nu v)\right),$$

where x, y are the point coordinates; u, v are the velocity vector components; $\vec{l_x}$, $\vec{l_y}$ are the unit vectors; $\nu = 0$ are the Cartesian coordinates, and $\nu = 1$ are the cylindrical coordinates. In what follows, we basically focus on the cylindrical coordinates.

Note that equation of internal energy (3) follows from the entropy equation

$$\frac{dE}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} = 0 \tag{6}$$

and equation of continuity (2), while equation of motion (1) and Eq. (3) give the equation of total energy

$$\frac{d}{dt}\left(E+\frac{\vec{u}^2}{2}\right)+\frac{1}{\rho}\nabla\cdot(P\vec{u})=0. \tag{7}$$

The solution of Eqs. (1)—(5) is sought in the domain $D=\{x_a<x<x_b,\ y_a<y<y_b\}$, the boundary of which is impermeable to the gas. Initial velocity, density and internal energy distributions are specified.

The above boundary-value problem is solved by the SPH method. For definiteness, consider the case of an axially symmetric gas flow ($\nu=1$) described in cylindrical coordinates (axial x and radial y).

Philosophy of the SPH method is expressed in two key formulas: approximate integral representation of functions and approximate calculation of integrals.

Let $A(\vec{r})$ be an arbitrary function defined and continuous over a domain D. According to the basic property of the δ-function, the following equality is valid:

$$A(\vec{r})=\iint A(\vec{r}^{\,*})\,\delta(\vec{r}-\vec{r}^{\,*})d\vec{r}^{\,*},\quad \vec{r}^{\,*},\vec{r}\in D. \tag{8}$$

Let us take a smooth function $W(\vec{r}-\vec{r}^{\,*},h)$, $\vec{r}$, $\vec{r}^{\,*}\in D$, $h>0$, called a kernel or smoothing function. It has a finite support defined by the smoothing length h. Suppose that the smoothing function possesses the following properties:

1) $\iint W(\vec{r}-\vec{r}^{\,*},h)\,d\vec{r}^{\,*}=1$;
2) $\lim\limits_{h\to 0} W(\vec{r}-\vec{r}^{\,*},h)=\delta(\vec{r}-\vec{r}^{\,*})$.

Here, convergence is understood to be the convergence in space of generalized functions.

Substituting the smoothing function for the δ-function in (8) gives the following approximate expression:

$$A(\vec{r})=\iint A(\vec{r}^{\,*})\,W(\vec{r}-\vec{r}^{\,*},h)\,d\vec{r}^{\,*}+O\left(h^2\right),\quad \vec{r}^{\,*},\vec{r}\in D. \tag{9}$$

As a specific smoothing function [6] we use a cubic spline of defect 1,

$$W=\frac{1}{a}\cdot\begin{cases}1-\dfrac{3}{2}R^2+\dfrac{3}{4}R^3, & 0\le R<1;\\ \dfrac{1}{4}(2-R)^3, & 1\le R\le 2;\\ 0, & 2<R,\end{cases}$$

where $R=|\vec{r}-\vec{r}^{\,*}|/h$; $a=1.5h$; $0.7\pi h^2$; πh^3 in the one-dimensional, two-dimensional and three-dimensional case, respectively. It is easy to prove that this function possesses the properties listed above.

To calculate the integral in (9), we divide the domain D (in the Oxy-plane) to subdomains D_j ($j=1,2,\ldots,N$). We call D_j a macroparticle (particle j) and denote the radius vector of its center of mass as $\vec{r}_j$. In the axially symmetric case, the particle is a solid of revolution. It has density ρ_j, internal energy E_j, axial u_j and radial v_j velocity components, and mass $m_j=2\pi y_j\sigma_j\rho_j$, where σ_j is the subdomain area. Note that the shape of the particle is not defined clearly.

Let us represent the integral as a sum of integrals over the subdomains:

$$\sum_{j=1}^{N}\iint\limits_{D_j}A(\vec{r}_j)\,W(\vec{r}-\vec{r}_j,h)\,\frac{2\pi y_j\rho_j}{2\pi y_j\rho_j}d\vec{r}_j\approx\frac{1}{2\pi}\sum_{j=1}^{N}\frac{m_j}{\rho_j y_j}A(\vec{r}_j)\,W(\vec{r}-\vec{r}_j,h). \tag{10}$$

Thus, formula (9) is reduced to the following approximate expression:

$$A(\vec{r})=\frac{1}{(2\pi)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_j y_j^\nu}A(\vec{r}_j)\,W(\vec{r}-\vec{r}_j,h). \tag{11}$$

It is the basic interpolation formula of the SPH method. It expresses the value of the scalar function A at the point $\vec{r}$ in terms of the values of A for the particles in the domain of the smoothing kernel and transfers differentiation from the unknown function A to the known function W. For example, the pressure gradient and velocity divergence can be expressed as

$$(\nabla P)_i = \frac{1}{(2\pi)^\nu} \sum_{j=1}^{N} \frac{m_j}{\rho_j y_j^\nu} P_j \nabla_i W_{ij}; \quad (\nabla \cdot \vec{u})_i = \frac{1}{(2\pi y_i)^\nu} \sum_{j=1}^{N} \frac{m_j}{\rho_j} \vec{u}_j \nabla_i W_{ij}.$$

The value $\nu = 0$ corresponds to Cartesian coordinates.

Note that the error in interpolation formula (11), as one can see from approximate equality (9), depends on the smoothing length, and, as suggested by transformation (10), on the number of particles falling into the circle $|\vec{r} - \vec{r}_j| < 2h$.

2. SPH equations to describe axially symmetric gas flow in cylindrical coordinates

2.1. Non-conservative system of gas-dynamic SPH equations. A system of ordinary differential gas-dynamic SPH equations must satisfy the laws of conservation of mass, momentum, and energy. In the SPH method, the law of conservation of mass is satisfied automatically, because continuum is modeled by a set of particles, the masses of which stay the same during motion. The two other laws require validation, but one should note that the law of conservation of the axial momentum component in the axially symmetric case must hold exactly.

For gas-dynamic equations, one can construct several systems of SPH equations using basic formula (11). Of special interest among them are fully conservative systems and the systems, which satisfy the Hamilton—Ostrogradsky variational principle of least action.

Note that the smoothing length in (11) is constant. In this section, we use this assumption for simplicity. In the subsequent sections, the smoothing length is assumed to be variable: $h = \frac{1}{2}|h_i - h_j|$.

Let us consider some SPH systems. Let us move from gas-dynamic equations (1)—(3) to those for particles by replacing the gradient and divergence by expressions given at the end of the previous section. This replacement gives a system of ordinary differential equations for unknown $\vec{u}_i$, ρ_i, E_i $(i = 1, 2, \ldots, N)$. It satisfies the law of conservation of energy. Indeed, putting together the equation of internal energy and the equation of motion multiplied by velocity, we obtain a conservative equation of energy for the i-th particle:

$$\frac{d}{dt}\left(E + \frac{\vec{u}^2}{2}\right)_i + \frac{1}{(2\pi)^\nu} \sum_{j=1}^{N} \frac{m_j}{\rho_i \rho_j} \left(\frac{P_i}{y_i^\nu}\vec{u}_j + \frac{P_j}{y_j^\nu}\vec{u}_i\right) \nabla_i W_{ij} = 0.$$

Multiplying this equation by m_i and taking a sum over all the particles, because of the antisymmetry of the kernel gradient, we ultimately have

$$\sum_{i=1}^{N} m_i \frac{d}{dt}\left(E + \frac{\vec{u}^2}{2}\right)_i = \frac{d}{dt} \sum_{i=1}^{N} m_i \left(E + \frac{\vec{u}^2}{2}\right)_i = 0.$$

Thus, energy is conserved, which is not the case for momentum. It is easy to demonstrate by simple examples that for the SPH equations under consideration, momentum is not conserved even in Cartesian coordinates. The reason for this violation is that there is no direct pairwise interaction between particles.

2.2. A near full conservative system of SPH equations. It follows from the analysis of gas-dynamic-to-SPH transformation (11) that one possibility for organizing the pairwise interaction is to first reduce the equations of motion and continuity to a form containing a density gradient. For this purpose, in [6], the authors develop basic gas-dynamic equations using the known formulas of vector analysis

$$\frac{1}{\rho}\nabla P = \nabla\left(\frac{P}{\rho}\right) + \frac{P}{\rho^2}\nabla(\rho); \qquad \rho \nabla \cdot \vec{u} = \nabla \cdot (\rho \vec{u}) - \vec{u} \cdot \nabla(\rho). \tag{12}$$

In cylindrical coordinates, interpolation formula (11) cannot represent the functions uniquely. For example, density can be defined in two ways:

$$1) \quad \rho_i = \frac{1}{2\pi} \sum_{j=1}^{N} \frac{m_j}{y_j} W_{ij}; \tag{13}$$

$$2) \quad \rho_i = \frac{1}{2\pi y_i} \sum_{j=1}^{N} m_j W_{ij}. \tag{14}$$

These representations, generally speaking, can lead to different systems of SPH equations.

Suppose the gas-dynamic equations are transformed in accordance with (12). Then, if we take density in the form of (13), from the gas-dynamic equations we obtain the following system of equations for the particles:

$$\begin{aligned}
&\frac{d\vec{u}_i}{dt} + \frac{1}{2\pi} \sum_{j=1}^{N} \frac{m_j}{y_j} \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij} \right) \nabla_i W_{ij} = 0; \\
&\frac{d\rho_i}{dt} + \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{\vec{u}_j}{y_i} - \frac{\vec{u}_i}{y_j} \right) \nabla_i W_{ij} = 0; \\
&\frac{dE_i}{dt} + \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2} + \frac{1}{2}\Pi_{ij} \right) \left(\frac{\vec{u}_j}{y_i} - \frac{\vec{u}_i}{y_j} \right) \nabla_i W_{ij} = 0 \qquad (i = 1, 2, \ldots, N).
\end{aligned} \tag{15}$$

The system contains an artificial viscosity term divided by squared density. It is composed of a linear part and a quadratic part with coefficients α and β, respectively:

$$\Pi_{ij} = \begin{cases} \dfrac{-\alpha c_{ij} \mu_{ij} + \beta \mu_{ij}^2}{\rho_{ij}}, & if \quad (\vec{u}_i - \vec{u}_j)(\vec{r}_i - \vec{r}_j) < 0; \\ 0, & if \quad (\vec{u}_i - \vec{u}_j)(\vec{r}_i - \vec{r}_j) \geq 0. \end{cases}$$

Here, $\mu_{ij} = h \dfrac{(\vec{u}_i - \vec{u}_j)(\vec{r}_i - \vec{r}_j)}{(\vec{r}_i - \vec{r}_j)^2 + (0.1h)^2}$; $c_{ij} = \dfrac{1}{2}(c_i + c_j)$, where c_i, c_j are the algebraic sound velocities; $\rho_{ij} = \dfrac{1}{2}(\rho_i + \rho_j)$.

Note that Eqs. (15) were used in [7] to model the Rayleigh—Taylor hydrodynamic instability.

Suppose that the SPH equations and Eq.(6) are subject to formal algebraic transformations. We call a system of gas-dynamic SPH equations fully conservative if:

1) the entropy equation $\left(\dfrac{dE_i}{dt} - \dfrac{P_i}{\rho_i^2} \dfrac{d\rho_i}{dt} = 0 \right)$ written for the i-th particle and the equation of continuity lead to the equation of internal energy;
2) the equations of motion and internal energy lead to the conservative equation of total energy;
3) the equation of motion leads to the law of variation of momentum.

In what follows, the first condition of conservativeness is called entropic. It implies that, in the equation of internal energy, pressure under the summation symbol is subtracted from (15) only in the i-th particle.

Along with the gas-dynamic SPH system one can consider its companion system. It preserves all the differential equations of the basic system, except for the equation of continuity; instead it uses an integral representation of density corresponding to the basic system. Let us call the companion system fully conservative, if the basic SPH system is fully conservative.

Note that system (15) in Cartesian coordinates (we remove $2\pi, y_i, y_j$) is fully conservative. Let us check whether system (15) satisfies the conditions of full conservativeness in cylindrical coordinates. Let us add entropy equation (6) for the i-th particle to system (15). Replacing the derivative of density with

respect to time in this equation by the derivative from the equation of continuity, we obtain the equation of internal energy. Then, adding it to the equation of motion multiplied by velocity gives the equation of total energy

$$\frac{d}{dt}\left(E+\frac{\vec{u}^2}{2}\right)_i+\frac{1}{2\pi}\sum_{j=1}^{N}m_j\left[\left(\frac{P_i}{\rho_i^2}+\frac{1}{2}\Pi_{ij}\right)\frac{\vec{u}_j}{y_i}+\left(\frac{P_j}{\rho_j^2}+\frac{1}{2}\Pi_{ij}\right)\frac{\vec{u}_i}{y_j}\right]\nabla_i W_{ij}=0,$$

having a conservative form, because the expression in the square brackets is symmetric. This equation results in the law of conservation of total energy. Thus, the two conditions of full conservativeness are satisfied. The third condition, the law of variation of momentum, is not satisfied. Indeed, generally speaking, the axial component of momentum is not conserved:

$$\sum_{i=1}^{N}m_i\vec{l}_x\frac{d\vec{u}_i}{dt}=-\sum_{i=1}^{N}m_i\vec{l}_x\left[\frac{1}{2\pi}\sum_{j=1}^{N}\frac{m_j}{y_j}\left(\frac{P_i}{\rho_i^2}+\frac{P_j}{\rho_j^2}+\Pi_{ij}\right)\nabla_i W_{ij}\right]\not\equiv 0.$$

Thus, the system of SPH equations (15) is not fully conservative, but it becomes near fully conservative at greater distances from the axis of symmetry.

2.3. First fully conservative system of SPH equations. Suppose density is now represented by (14). The density gradient is then calculated by

$$\nabla\rho_i=\frac{1}{2\pi y_i}\sum_{j=1}^{N}m_j\nabla_i W_{ij}-\frac{\rho_i}{y_i}\nabla y_i, \tag{16}$$

and the system of SPH equations for two-dimensional axially symmetric flows can be written as

$$\begin{aligned}
&\frac{d\vec{u}_i}{dt}+\frac{1}{2\pi}\sum_{j=1}^{N}m_j\left(\frac{P_i}{\rho_i^2 y_i}+\frac{P_j}{\rho_j^2 y_j}+\Pi_{ij}\right)\nabla_i W_{ij}=\frac{P_i}{\rho_i y_i}\nabla y_i;\\
&\frac{d\rho_i}{dt}+\frac{1}{2\pi y_i}\sum_{j=1}^{N}m_j\left(\vec{u}_j-\vec{u}_i\right)\nabla_i W_{ij}=-\vec{u}_i\frac{\rho_i}{y_i}\nabla y_i;\\
&\frac{dE_i}{dt}+\frac{1}{2\pi}\sum_{j=1}^{N}m_j\left(\frac{P_i}{\rho_i^2 y_i}+\frac{1}{2}\Pi_{ij}\right)\left(\vec{u}_j-\vec{u}_i\right)\nabla_i W_{ij}=-\frac{P_i}{\rho_i y_i}\vec{u}_i\nabla y_i\quad(i=1,2,\ldots,N).
\end{aligned} \tag{17}$$

The artificial viscosity in these equations slightly differs from the viscosity in (15); in particular, instead of ρ_{ij} it uses $\rho_{ij}=\left(\rho_i y_i+\rho_j y_j\right)/2$.

Written in a different form, the system of SPH equations (17) is given in [8]. In that work, the equation of motion is obtained by the variational method based on the Hamilton—Ostrogradksy principle of least action and shown to be derivable by formal transformation of Eq. (1) with subsequent use of interpolation formula (11) of the SPH method.

The system is remarkable because it satisfies all the conditions of full conservativeness. Indeed, the equation following from entropy equation (6) and equation of continuity from (17), after adding the viscosity to it, coincides with the equation of internal energy in (17). Now, from the equations of motion and internal energy it is easy to obtain the conservative equation of total energy

$$\frac{d}{dt}\left(E+\frac{\vec{u}^2}{2}\right)_i+\frac{1}{2\pi}\sum_{j=1}^{N}m_j\left[\left(\frac{P_i}{\rho_i^2 y_i}+\frac{1}{2}\Pi_{ij}\right)\vec{u}_j+\left(\frac{P_j}{\rho_j^2 y_j}+\frac{1}{2}\Pi_{ij}\right)\vec{u}_i\right]\nabla_i W_{ij}=0.$$

Thus, considering the equations of motion and continuity, we have algebraically equivalent equations of total energy, internal energy, and conservation of entropy. Finally, by multiplying the equation of motion

by the mass of particle, and taking a total over all particles, we obtain the following series of equalities:

$$\sum_{i=1}^{N} m_i \frac{d\vec{u}_i}{dt} = \sum_{i=1}^{N} m_i \left[\frac{P_i}{\rho_i y_i} \nabla y_i - \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2 y_i} + \frac{P_j}{\rho_j^2 y_j} + \Pi_{ij} \right) \nabla_i W_{ij} \right] = \sum_{i=1}^{N} m_i \frac{P_i}{\rho_i y_i} \nabla y_i.$$

From this we have the following law of variation of momentum for an assembly of particles:

$$\frac{d}{dt} \sum_{i=1}^{N} m_i \vec{u}_i = \sum_{i=1}^{N} m_i \frac{P_i}{\rho_i y_i} \nabla y_i,$$

from which it follows that the axial component of momentum is conserved, while the radial one varies in accordance with the acting force. Thus, the system of SPH equations (17) is fully conservative.

Note that system (17) could be obtained without direct use of (16). To get that done, it was sufficient to replace formulas (12) in the transformation of the gas-dynamic equations by identical relations

$$\frac{1}{\rho} \nabla P = \nabla \left(\frac{P}{\rho} \right) + \frac{P}{\rho^2} \nabla (y\rho) - \frac{P}{\rho y} \nabla y; \qquad \rho \nabla \cdot \vec{u} = \nabla \cdot (\rho \vec{u}) - \frac{1}{y} \vec{u} \cdot \nabla (y\rho) + \frac{\rho}{y} \vec{u} \cdot \nabla y.$$

Interestingly, in this case, representing the density as (13) results in the same system (17). However, the use of two different density representations with different settings of the basic gas-dynamic equations gives two different ((15) and (17)) SPH systems. The first is not conservative with respect to the axial component of momentum, while the second is conservative.

2.4. Second fully conservative system of SPH equations. Note that for the density given by (13) one can construct another SPH system, if for the transformation of the basic equations we use identity relations

$$\frac{1}{\rho} \nabla P = \frac{1}{y} \nabla \left(y \frac{P}{\rho} \right) + \frac{P}{\rho^2} \nabla \rho - \frac{P}{\rho y} \nabla y; \qquad \rho \nabla \cdot \vec{u} = \widetilde{\nabla} \cdot (\rho \vec{u}) - \vec{u} \cdot \nabla \rho + \frac{\rho}{y} \vec{u} \cdot \nabla y,$$

where

$$\widetilde{\nabla} \cdot (\rho \vec{u}) = \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}.$$

The only difference of this system from (17) is that index i at the radial variable under the summation symbol is changed to index j, and, vice versa, index i is used instead of j:

$$\begin{aligned}
&\frac{d\vec{u}_i}{dt} + \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2 y_j} + \frac{P_j}{\rho_j^2 y_i} + \Pi_{ij} \right) \nabla_i W_{ij} = \frac{P_i}{\rho_i y_i} \nabla y_i; \\
&\frac{d\rho_i}{dt} + \frac{1}{2\pi} \sum_{j=1}^{N} \frac{m_j}{y_j} (\vec{u}_j - \vec{u}_i) \nabla_i W_{ij} = -\vec{u}_i \frac{\rho_i}{y_i} \nabla y_i; \\
&\frac{dE_i}{dt} + \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2 y_j} + \frac{1}{2} \Pi_{ij} \right) (\vec{u}_j - \vec{u}_i) \nabla_i W_{ij} = -\frac{P_i}{\rho_i y_i} \vec{u}_i \nabla y_i \quad (i = 1, 2, \ldots, N).
\end{aligned} \tag{18}$$

System (18) is fully conservative. It satisfies all the conditions of full conservativeness; in particular, the equation of total energy is given by

$$\frac{d}{dt} \left(E + \frac{\vec{u}^2}{2} \right)_i + \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left[\left(\frac{P_i}{\rho_i^2 y_j} + \frac{1}{2} \Pi_{ij} \right) \vec{u}_j + \left(\frac{P_j}{\rho_j^2 y_i} + \frac{1}{2} \Pi_{ij} \right) \vec{u}_i \right] \nabla_i W_{ij} = 0.$$

Let us show that the equation of motion in (18) can be derived from the Hamilton—Ostrogradksy variational principle of least action.

Consider the motion of an assembly of particles. Denote the kinetic energy by $T = \sum\limits_{j=1}^{N} m_j \dfrac{\vec{u}_j^2}{2}$ and the internal energy by $U = \sum\limits_{j=1}^{N} m_j E_j(\rho_j, S_j)$. The function $L = T - U$ is called the Lagrangian function, and the functional $\int\limits_{t_0}^{t_1} L dt$ is called the action.

According to the Hamilton—Ostrogradksy principle, true motion of particles during the time interval (t_0, t_1) is described by the functions $x_i(t)$, $y_i(t)$, providing a minimum for the action functional. If this minimum is reached, then the Euler—Lagrange equations hold true:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right) - \frac{\partial L}{\partial x_i} = 0; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial y_i} = 0 \quad (i = 1, 2, \ldots, N).$$

Suppose that the gas motion is isentropic. Then, since $\dfrac{\partial E}{\partial \rho} = \dfrac{P}{\rho^2}$ according to the first law of thermodynamics, the Euler—Lagrange equations can be reduced to

$$m_i \frac{du_i}{dt} + \sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2} \frac{\partial \rho_j}{\partial x_i} = 0; \quad m_i \frac{dv_i}{dt} + \sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2} \frac{\partial \rho_j}{\partial y_i} = 0 \quad (i = 1, 2, \ldots, N).$$

Using density representation (13), in this system of equations we replace density differentiation by differentiation of the interpolation kernel. Let us transform the group of equations corresponding to the radial direction of motion:

$$\begin{aligned}
-m_i \frac{dv_i}{dt} &= \sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2} \frac{\partial \rho_j}{\partial y_i} = m_i \frac{P_i}{\rho_i^2} \frac{\partial \rho_i}{\partial y_i} + \sum_{\substack{j=1,\\ j\neq i}}^{N} m_j \frac{P_j}{\rho_j^2} \frac{\partial \rho_j}{\partial y_i} = \\
&= -\frac{m_i}{y_i^2} \frac{1}{2\pi} \sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2} W_{ij} + m_i \frac{1}{2\pi} \left(\frac{P_i}{\rho_i^2} \sum_{\substack{j=1,\\ j\neq i}}^{N} \frac{m_j}{y_j} \frac{\partial W_{ij}}{\partial y_i} + \sum_{\substack{j=1,\\ j\neq i}}^{N} m_j \frac{P_j}{\rho_j^2} \frac{1}{y_j} \frac{\partial W_{ij}}{\partial y_i} \right) = \\
&= -m_i \left(\frac{P}{\rho y} \right)_i + m_i \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2 y_j} + \frac{P_j}{\rho_j^2 y_i} \right) \frac{\partial W_{ij}}{\partial y_i}.
\end{aligned}$$

The equations for the axial direction are transformed identically:

$$\begin{aligned}
-m_i \frac{du_i}{dt} &= \sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2} \frac{\partial \rho_j}{\partial x_i} = \; m_i \frac{P_i}{\rho_i^2} \frac{1}{2\pi} \sum_{j=1}^{N} \frac{m_j}{y_j} \frac{\partial W_{ij}}{\partial x_i} + \\
&+ \sum_{\substack{j=1,\\ j\neq i}}^{N} m_j \frac{P_j}{\rho_j^2} \frac{\partial}{\partial x_i} \left(\frac{1}{2\pi} \sum_{k=1}^{N} \frac{m_k}{y_k} W_{jk} \right) = m_i \frac{1}{2\pi} \sum_{j=1}^{N} m_j \left(\frac{P_i}{\rho_i^2 y_j} + \frac{P_j}{\rho_j^2 y_i} \right) \frac{\partial W_{ij}}{\partial x_i}.
\end{aligned}$$

By equating the first term to the last one in these series of equations, and by canceling m_i and adding the artificial viscosity, we obtain the equation of motion in (18).

2.5. Third fully conservative system of SPH equations. Let us present one more system of SPH equations. To organize pairwise particle interaction, it uses a fundamentally different approach. The latter hypothesizes that the constant is reproduced with sufficient accuracy by the interpolation formula (11). Let us introduce two identities:

$$\nabla(aP) = \frac{a}{y^\nu} \nabla(y^\nu P) + P \nabla a - \nu a \frac{P}{y} \nabla y; \quad \nabla \cdot \vec{u} = \widetilde{\nabla} \cdot \vec{u} - \vec{u} \nabla a + \nu \vec{u} \frac{\nabla y}{y},$$

where $\nu = 0\,or\,1$; $a \equiv 1$; $\widetilde{\nabla} \cdot \vec{u} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$.

Let us use these identities for preliminary transformation of the gas-dynamic equations and then apply formula (11). This gives the following system of SPH equations:

$$
\begin{aligned}
&\frac{d\vec{u}_i}{dt} + \frac{1}{(2\pi)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_i\rho_j}\left(\frac{P_i}{y_j^\nu} + \frac{P_j}{y_i^\nu} + \Omega_{ij}\right)\nabla_i W_{ij} = \nu\frac{P_i}{\rho_i y_i}\nabla y_i;\\
&\frac{d\rho_i}{dt} + \frac{\rho_i}{(2\pi)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_j}\frac{1}{y_j^\nu}\left(\vec{u}_j - \vec{u}_i\right)\nabla_i W_{ij} = -\nu\vec{u}_i\frac{\rho_i}{y_i}\nabla y_i;\\
&\frac{dE_i}{dt} + \frac{1}{(2\pi)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_i\rho_j}\left(\frac{P_i}{y_j^\nu} + \frac{1}{2}\Omega_{ij}\right)\left(\vec{u}_j - \vec{u}_i\right)\nabla_i W_{ij} = -\nu\frac{P_i}{\rho_i y_i}\vec{u}_i\nabla y_i \quad (i = 1, 2, \ldots, N)\,.
\end{aligned}
\tag{19}
$$

Here, $\Omega_{ij} = \rho_{ij}^2\dfrac{1}{y_{ij}^\nu}\Pi_{ij}$; $y_{ij}^\nu = \dfrac{1}{2}\left(y_i^\nu + y_j^\nu\right)$; Π_{ij} is from (15). For $\nu = 1$, the system of equations (19) is given in a slightly different form in Ref. [9] devoted to SPH numerical simulations of shaped charge performance. It is easy to show that system (19) satisfies all the conditions of full conservativeness. In particular, from (19) follows the conservative equation of total energy

$$
\frac{d}{dt}\left(E + \frac{\vec{u}^2}{2}\right)_i + \frac{1}{(2\pi)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_i\rho_j}\left[\left(\frac{P_i}{y_j^\nu} + \frac{1}{2}\Omega_{ij}\right)\vec{u}_j + \left(\frac{P_j}{y_i^\nu} + \frac{1}{2}\Omega_{ij}\right)\vec{u}_i\right]\nabla_i W_{ij} = 0.
$$

Thus, several systems of SPH equations are available for solving axially symmetric gas-dynamic problems:
1) (15); 2) (17); 3) (18); 4) (19).

The first system on this list is not fully conservative. It violates the law of variation of momentum (the axial component of momentum is not conserved). In Cartesian coordinates, however, all the four systems are conservative; interestingly, the first three systems coincide. The equation of motion in the second and third SPH systems can be obtained by the variational method. This is the main difference between them and the fourth system. Therefore, when choosing the most appropriate system, preference should apparently be given to SPH system (17) or system (18).

Comment. In a region far from the axis of symmetry, ($y_i \gg h_i$), SPH system (15) is virtually near fully conservative.

2.6. Five other systems of gas-dynamic SPH equations. It is evident that the list of SPH systems can be continued using other representations for the gradient and divergence in the basic equations. For example, using additional identity transformations

$$
\begin{aligned}
&\frac{P}{\rho}\nabla\cdot\vec{u} = \nabla\cdot\left(\frac{P}{\rho}\vec{u}\right) - \vec{u}\cdot\nabla\left(\frac{P}{\rho}\right);\\
&\frac{P}{\rho}\nabla\cdot\vec{u} = \widetilde{\nabla}\cdot\left(\frac{P}{\rho}\vec{u}\right) - \vec{u}\cdot\nabla\left(\frac{P}{\rho}\right) + \frac{P}{\rho y}\vec{u}\,\nabla y;\\
&\frac{P}{\rho}\nabla\cdot\vec{u} = \nabla\cdot\left(\frac{P}{\rho}\vec{u}\right) - \frac{1}{y}\vec{u}\cdot\nabla\left(y\frac{P}{\rho}\right) + \frac{P}{\rho y}\vec{u}\,\nabla y;\\
&\frac{P}{\rho}\nabla\cdot\vec{u} = \frac{P}{\rho}\widetilde{\nabla}\cdot\vec{u} + \nu\frac{P}{\rho y}\vec{u}\,\nabla y
\end{aligned}
$$

for the first, second, third, and fourth systems, respectively, it is easy to construct their variants, which instead of the equation of internal energy use the following equations, respectively:

$$5)\quad \frac{dE_i}{dt}+\frac{1}{4\pi}\sum_{j=1}^{N}m_j\left(\frac{P_i}{\rho_i^2}+\frac{P_j}{\rho_j^2}+\Pi_{ij}\right)\left(\frac{\vec{u}_j}{y_i}-\frac{\vec{u}_i}{y_j}\right)\nabla_i W_{ij}=0;$$

$$6)\quad \frac{dE_i}{dt}+\frac{1}{4\pi}\sum_{j=1}^{N}m_j\left(\frac{P_i}{\rho_i^2 y_i}+\frac{P_j}{\rho_j^2 y_j}+\Pi_{ij}\right)(\vec{u}_j-\vec{u}_i)\nabla_i W_{ij}=-\frac{P_i}{\rho_i y_i}\vec{u}_i\nabla y_i;$$

$$7)\quad \frac{dE_i}{dt}+\frac{1}{4\pi}\sum_{j=1}^{N}m_j\left(\frac{P_i}{\rho_i^2 y_j}+\frac{P_j}{\rho_j^2 y_i}+\Pi_{ij}\right)(\vec{u}_j-\vec{u}_i)\nabla_i W_{ij}=-\frac{P_i}{\rho_i y_i}\vec{u}_i\nabla y_i;$$

$$8)\quad \frac{dE_i}{dt}+\frac{1}{2}\frac{1}{(2\pi)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_i\rho_j}\left(\frac{P_i}{y_j^\nu}+\frac{P_j}{y_i^\nu}+\Omega_{ij}\right)(\vec{u}_j-\vec{u}_i)\nabla_i W_{ij}=-\nu\frac{P_i}{\rho_i y_i}\vec{u}_i\nabla y_i.$$

Compared to systems 1—4 above, their variants, systems 5—8, are less conservative. Indeed, for systems 5—8, the entropy condition is not satisfied, because the above equations of internal energy do not follow directly from entropy equation (6) and the equations of continuity in systems 1—4. Thus, conservative systems 6—8 are not fully conservative, and non-conservative system 5 does not tend to full conservativeness at greater distances from the axis of symmetry, because it is not fully conservative in Cartesian coordinates.

It is easy to construct an SPH system satisfying the entropy condition and the law of conservation of momentum but violates the law of conservation of total energy, like the following gas-dynamic SPH system:

$$\frac{d\vec{u}_i}{dt}+\frac{1}{(2\pi)^\nu}\sum_{j=1}^{N}m_j\left(\frac{P_i}{\rho_i^2 y_i^\nu}+\frac{P_j}{\rho_j^2 y_j^\nu}\right)\nabla_i W_{ij}=\nu\frac{P_i}{\rho_i y_i}\nabla y_i;$$

$$\frac{d\rho_i}{dt}+\frac{\rho_i}{(2\pi y_i)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_j}\vec{u}_j\nabla_i W_{ij}=0;$$

$$\frac{dE_i}{dt}+\frac{1}{(2\pi y_i)^\nu}\sum_{j=1}^{N}\frac{m_j}{\rho_i\rho_j}P_i\vec{u}_j\nabla_i W_{ij}=0.$$

In this system, the equation of motion is the same as in (17), and the equations of continuity and internal energy are taken from the non-conservative system in Subsection 2.1.

3. Integration of the gas-dynamic SPH system

3.1. Fully conservative cell-centered difference scheme. Let us explain the idea of constructing a fully conservative integration scheme for SPH equations using the case of a simple system of difference equations being a second-order approximation of plane-geometry Lagrangian one-dimensional gas-dynamic equations on smooth solutions and uniform mesh:

$$\frac{\partial u}{\partial t}+\frac{\partial\,(P+q)}{\partial m}=0;\quad \frac{\partial x}{\partial t}=u;\quad \frac{\partial V}{\partial t}-\frac{\partial u}{\partial m}=0;\quad \frac{\partial E}{\partial t}+(P+q)\,\frac{\partial u}{\partial m}=0.$$

Here, m is the Lagrangian mass coordinate; q is the artificial viscosity [10]; $V=1/\rho$ is the specific volume. Equation of state (5) is taken in the form of $P=P\,(\rho,E)$.

The second equation in the system is equation (4) describing the motion of material points (particles). Entropy equation (6) and equation of total energy (7) in Lagrangian coordinates are given by

$$\frac{\partial E}{\partial t} + P\frac{\partial V}{\partial t} = 0; \qquad \frac{\partial \left(E + \dfrac{u^2}{2}\right)}{\partial t} + \frac{\partial (P+q)u}{\partial m} = 0.$$

The difference scheme to determine pressure at the half-integer level is constructed using an auxiliary equation

$$\frac{\partial P}{\partial t} + a^2 \frac{\partial u}{\partial m} = 0,$$

where $a^2 = \rho^2 \dfrac{\partial P}{\partial \rho} + (P+q)\dfrac{\partial P}{\partial E}$. As one can easily see, on smooth solutions it follows from the two last equations in the above system. Note that for zero pseudo-viscosity $a = \rho c$.

Let us introduce a difference mesh, (m_k, t^n) $(k = 1, 2, \ldots;\ n = 0, 1, 2, \ldots)$. It is used to consider mesh functions $f(m_k, t^n) \equiv f_k^n$. To write difference equations, we use the following notation:

$$\tau = t^{n+1} - t^n; \quad r_k = \frac{\tau}{\Delta m_k}; \quad \Delta f_k = f_{k+1/2} - f_{k-1/2}; \quad \bar{f}_k = 0.5\left(f_k^{n+1} + f_k^n\right);$$
$$P_{k+1/2} = \frac{P_k \Delta x_k + P_{k+1}\Delta x_{k+1}}{\Delta x_k + \Delta x_{k+1}}; \quad u_{k+1/2} = \frac{u_k \Delta x_{k+1} + u_{k+1}\Delta x_k}{\Delta x_k + \Delta x_{k+1}};$$
$$\Delta x_k = 0.5\left(x_{k+1} - x_{k-1}\right); \quad \Delta m_k = m_{k+1/2} - m_{k-1/2}.$$

The gas-dynamic equations are integrated by the modified *predictor—corrector* method. For the time interval from t^n to t^{n+1}, the solution is constructed in two steps. First, at the predictor step, pressure is determined at an intermediate level. Pressure can be found in two ways: from the equation of state

$$P_k^{n+1/2} = P\left(\rho_k^{n+1/2}, E_k^{n+1/2}\right),$$

where

$$E_k^{n+1/2} - E_k^n + 0.5(P_k^n + q_k^n) r_k \Delta u_k^n = 0;$$
$$V_k^{n+1/2} - V_k^n - 0.5 r_k \Delta u_k^n = 0;$$
$$x_k^{n+1/2} = x_k^n + \tau u_k^n,$$

or from the auxiliary equation

$$P_k^{n+1/2} - P_k^n + 0.5 a_k^2 r_k \Delta u_k^n = 0.$$

Next, at the corrector step, the quantities are refined at the upper time level:

$$u_k^{n+1} - u_k^n + r_k \Delta (P_k^{n+1/2} + q_k^n) = 0;$$
$$E_k^{n+1} - E_k^n + (P_k^{n+1/2} + q_k^n) r_k \Delta \bar{u}_k = 0;$$
$$V_k^{n+1} - V_k^n - r_k \Delta \bar{u}_k = 0;$$
$$x_k^{n+1} = x_k^n + \tau \bar{u}_k.$$

The difference scheme is fully conservative. Indeed, the cell mass is conserved within the Lagrangian approach by construction, the law of conservation of momentum in the planar case is satisfied exactly, and the difference equation of internal energy follows directly from the difference equations of entropy and conservation of volume. Finally, multiplying the difference equation of motion by $\bar{u}_k$ and adding it to the difference equation of internal energy, we obtain a difference equation in the conservative form

$$\left(E + \frac{u^2}{2}\right)_k^{n+1} - \left(E + \frac{u^2}{2}\right)_k^n + r_k\left((Pu)_{k+1/2} - (Pu)_{k-1/2}\right) = 0,$$

where $(Pu)_{k+1/2} = \dfrac{\left(P_k^{n+1/2} + q_k^n\right)\bar{u}_{k+1}\Delta x_k + \left(P_{k+1}^{n+1/2} + q_{k+1}^n\right)\bar{u}_k\Delta x_{k+1}}{\Delta x_k + \Delta x_{k+1}}$.

A Fourier stability analysis of the system of difference equations in the acoustic approximation with "frozen" coefficients and mass-uniform mesh leads to the following necessary spectral von Neumann stability criterion :

$$\frac{a\tau}{\Delta m} \le 2.$$

3.2. Integration scheme for the gas-dynamic SPH equations. All the above SPH systems are integrated by the modified *predictor—corrector* method, in which, at the second step, the intermediate-level velocity is defined as an arithmetic mean of the upper- and lower-level velocities. This makes the method more stable and enables correct calculations of kinetic energy variations.

Following the above algorithm of constructing a fully conservative difference scheme, let use describe the method of integration of SPH system (17). At this point, we do not discuss the resolution of boundary conditions, because this issue requires special consideration. Note that this issue in the boundary condition settings for the problem stated in Sect. 1 is resolved relatively easily by means of algorithms, by which particles are reflected from rigid walls.

Let us introduce the following notation: $A = \dfrac{P}{\rho^2 y}$.

First step (predictor). Calculate particle coordinates, internal energy, density, and pressure in each particle ($i = 1, 2, \ldots, N$) at the intermediate level:

$$\begin{aligned}
&\vec{r}_i^{\,n+1/2} = \vec{r}_i^{\,n} + \frac{\tau}{2}\vec{u}_i^{\,n};\\
&E_i^{n+1/2} = E_i^n + \frac{\tau}{2}\left[-\left(\frac{Pv}{\rho y}\right)_i^n + \frac{1}{2\pi}\sum_j m_j\left(A_i + \frac{1}{2}\Pi_{ij}\right)^n (\vec{u}_i - \vec{u}_j)^n\,\nabla_i W_{ij}^n\right];\\
&\rho_i^{n+1/2} = \rho_i^n + \frac{\tau}{2}\left[-\left(\frac{\rho v}{y}\right)_i^n + \frac{1}{2\pi y_i^n}\sum_j m_j\,(\vec{u}_i - \vec{u}_j)^n\,\nabla_i W_{ij}^n\right];\\
&P_i^{n+1/2} = P\left(\rho_i^{n+1/2}, E_i^{n+1/2}\right).
\end{aligned} \tag{20}$$

Second step (corrector). The second step consists of two substeps. First, particle velocities are calculated at the upper and intermediate levels, and then, internal energy, density, and coordinates of each particle are determined:

1) $i = 1, 2, \ldots, N$;

$$\begin{aligned}
&u_i^{n+1} = u_i^n - \frac{\tau}{2\pi}\sum_j m_j\left(A_i^{n+1/2} + A_j^{n+1/2} + \Pi_{ij}^n\right)\frac{\partial}{\partial x_i}\left(W_{ij}^{n+1/2}\right);\\
&v_i^{n+1} = v_i^n - \frac{\tau}{2\pi}\sum_j m_j\left(A_i^{n+1/2} + A_j^{n+1/2} + \Pi_{ij}^n\right)\frac{\partial}{\partial y_i}\left(W_{ij}^{n+1/2}\right) + \tau\left(\frac{P}{\rho y}\right)_i^{n+1/2};\\
&\vec{u}_i^{\,n+1/2} = \frac{1}{2}\left(\vec{u}_i^{\,n} + \vec{u}_i^{\,n+1}\right);
\end{aligned}$$

2) $i = 1, 2, \ldots, N$;

$$E_i^{n+1} = E_i^n + \tau \left[-\left(\frac{Pv}{\rho y}\right)_i^{n+1/2} + \frac{1}{2\pi} \sum_j m_j \left(A_i^{n+1/2} + \frac{1}{2}\Pi_{ij}^n \right) (\vec{u}_i - \vec{u}_j)^{n+1/2} \nabla_i W_{ij}^{n+1/2} \right];$$

$$\rho_i^{n+1} = \rho_i^n + \tau \left[-\left(\frac{\rho v}{y}\right)_i^{n+1/2} + \frac{1}{2\pi y_i^{n+1/2}} \sum_j m_j (\vec{u}_i - \vec{u}_j)^{n+1/2} \nabla_i W_{ij}^{n+1/2} \right]; \qquad (21)$$

$$\vec{r}_i^{n+1} = \vec{r}_i^n + \tau \vec{u}_i^{n+1/2}.$$

The integration scheme is fully conservative. Indeed, the equation of internal energy follows from the difference entropy equation

$$\frac{E_i^{n+1} - E_i^n}{\tau} - \frac{P_i^{n+1/2}}{\left(\rho_i^{n+1/2}\right)^2} \frac{\rho_i^{n+1} - \rho_i^n}{\tau} = 0$$

and equation of continuity, so the entropy condition is satisfied. By multiplying the equation of motion by the particle mass and taking a sum over all particles, we make sure that the law of conservation of momentum is observed. Finally, from the equations of motion and internal energy follows the fully conservative equation of total energy

$$(EP)_i^{n+1} - (EP)_i^n + \frac{\tau}{2\pi} \sum_j m_j \left[\left(A_j^{n+1/2} + \frac{1}{2}\Pi_{ij}^n \right) \vec{u}_i^{n+1/2} + \left(A_i^{n+1/2} + \frac{1}{2}\Pi_{ij}^n \right) \vec{u}_j^{n+1/2} \right] \nabla_i W_{ij}^{n+1/2} = 0,$$

where $EP = E + \vec{u}^2/2$.

Comment 1. The SPH method admits density calculations using both differential (20), (21), and integral (14) density representations. Therefore, the following algorithm can be used:

1) at the first step, instead of (20) we use

$$\rho_i^{n+1/2} = \frac{1}{2\pi y_i^{n+1/2}} \sum_{j=1}^N m_j W_{ij}^{n+1/2};$$

2) at the second step, instead of (21) we take

$$\rho_i^{n+1} = \frac{1}{2\pi y_i^{n+1}} \sum_{j=1}^N m_j W_{ij}^{n+1},$$

i. e. density is completely defined from the integral representation.

Comment 2. If SPH system (15) or (18) is used to describe an axially symmetric flow, then as an integral density representation we use (13).

4. Numerical example

Consider a one-dimensional spherical problem of a strong blast [11]. The initial problem geometry consists of two regions: the first is a sphere of radius $r = 0.1$ with a center at the origin and energy $E_0 = 10^7$. The second region is cylindrical, with zero energy. The gas is at rest, its density is $\rho = 1$, and the adiabatic index is $\gamma = 1.4$.

The problem is axially symmetric up to the time $t = 0.75$. The current analytical location of the shock front is $r_f = 7.737$; the shock density is $\rho_f = 6$, the pressure is $p_f = 14.189$, and the velocity is

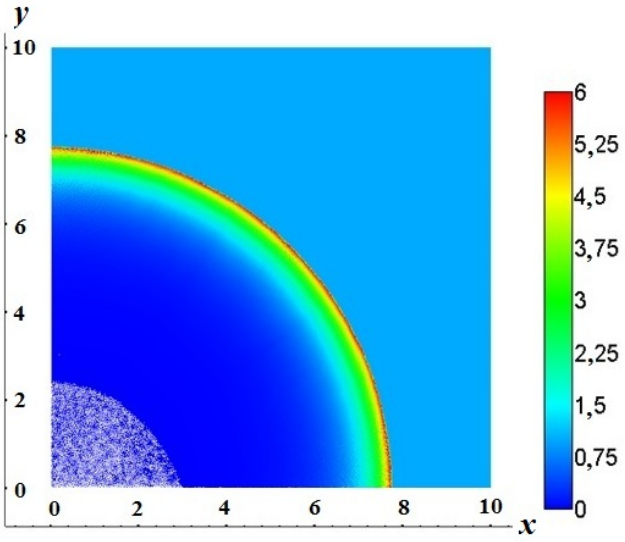

Fig. 1. Density distribution

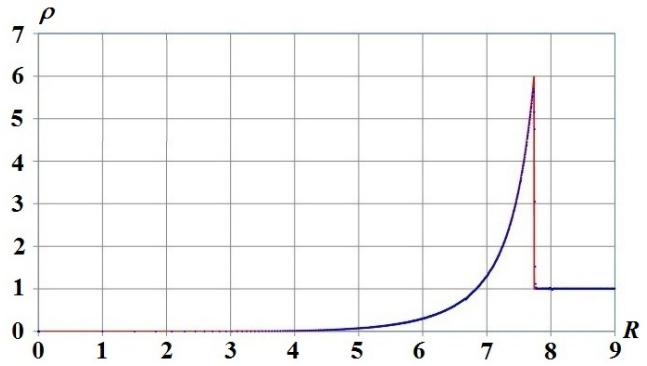

Fig. 2. Density as a function of radius: — is the exact solution; • is the calculation

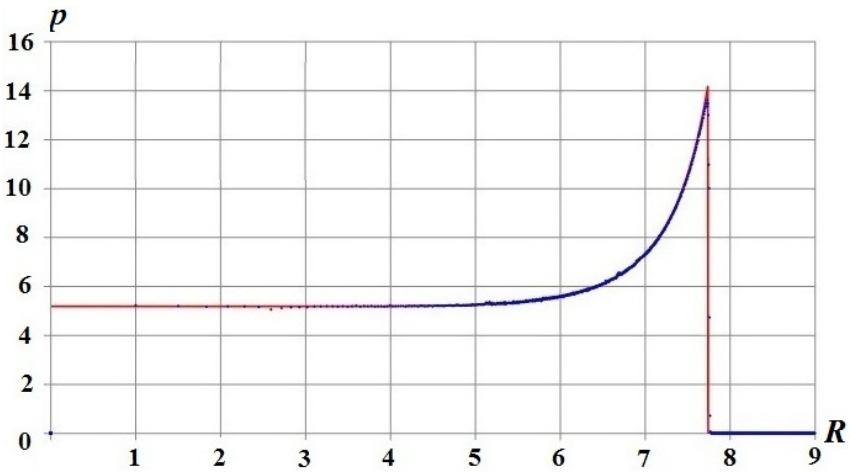

Fig. 3. Pressure as a function of radius: — is the exact solution; • is the calculation

$u_f = 3.439$ [11]. A special algorithm is used on the axis to enable transition to three-dimensional problem calculations.

The domain is bounded by a cylinder $0 \le x,\ y \le 10$ divided into six spherical and one outer layer for particle generation. The pitch between particles increases 1.5 times from layer to layer. In the first layer corresponding to the region with specified blast energy, particles are placed with a pitch of 0.00087. The total number of particles in the problem is 1.8 million.

Fig. 1 shows the final density distribution.

Figs. 2—4 show the angle-average plots of density, pressure, and velocity as a function of final radius. One can see close agreement with the analytical solution. The energy imbalance in the calculation did not exceed 0.013 %. The reason for the imbalance is the imperfection of the mechanism of transition from the two-dimensional axially symmetric problem to three dimensions in Cartesian coordinates in the region near the axis of rotation. In transition-free calculations, the imbalance was comparable to computational errors. For example, in the planar cylindrical blast problem, the imbalance at the time when the shock traveled to a distance of 60 times the initial radius was $2.2 \cdot 10^{-11}$ %. The initial energy, density, equation of state, number of particles, and their initial pitch were the same as in the above spherical problem.

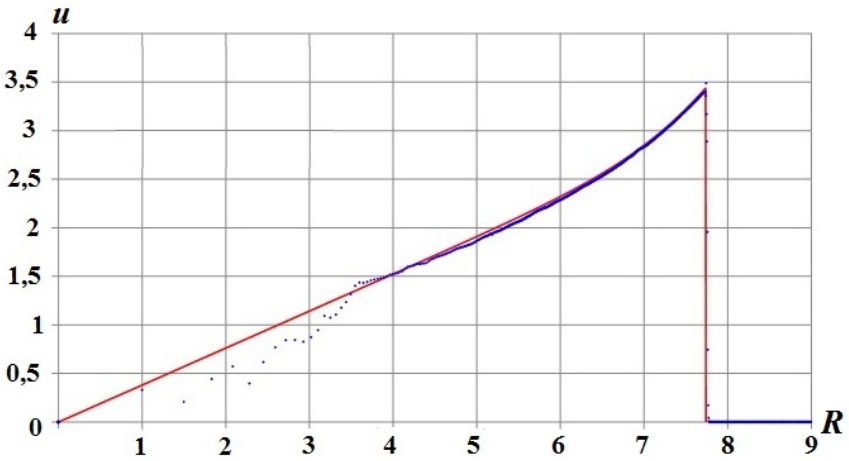

Fig. 4. Velocity as a function of radius: —— is the exact solution; • is the calculation

Conclusion

Let us summarize the results of this work:

1. The formalism of the SPH method allows us to construct several different systems of SPH equations for gas-dynamic equations. In this work, we give a definition of fully conservative systems of gas-dynamic SPH equations, and, among all the systems, identify the fully conservative ones. We present ten systems, four of which are not conservative (including two, which are not conservative even in Cartesian coordinates), three of which satisfy the property of full conservativeness, and the rest of which are conservative, but not to the full extent.
2. Two of the fully conservative systems are known; they have been published in [8, 9]. In the third fully conservative system, exactly as in Brookshow [8], the equation of motion is constructed by the variational method using the Hamilton—Ostrogradksy principle of least action. The new system is nearly the same as the system of Brookshow [8]; the differences are due to different density representations: as an average of "planar" densities, or as a space average.
3. A modified *predictor—corrector* method is proposed for numerical integration of the systems of SPH equations. The method is based on the fully conservative cell-centered difference scheme of Lagrangian gas dynamics. The modified method preserves the full conservativeness of the initial SPH system. In the paper, we construct an integration scheme for the SPH equations of Brookshow [8]; for the other systems, the modified method is implemented identically.
4. The possibility of using the fully conservative integration scheme for numerical modeling of two-dimensional axially symmetric gas flows is demonstrated through calculations of a strong spherical blast problem.

References

1. *Gingold R. A., Monaghan J. J.* Smoothed particle hydrodynamics: theory and application to non-spherical stars // Mon. Not. Roy. Astron. Soc. 1977. No 181. P. 375.
2. *Lucy L. B.* A numerical approach to the testing of the fission hypothesis // Astron. J. 1977. Vol. 82, No 12. P. 1013.
3. *Popov Yu. P., Samarskiy A. A.* Fully conservative difference schemes // Journal of Computational Mathematics and Mathematical Physics. 1969. V. 9, No. 4. Pp. 953—958. (In Russian)
Popov Yu. P., Samarskiy A. A. Polnostyu konservativnye raznostnye skhemy // Zhurnal vychisl. mat. i mat. fiz. 1969. V. 9, No. 4. S. 953—958.
4. *Harlow F. H.* The particle-in-cell computing method for fluid dynamics // Computational Methods in Hydrodynamics. Moscow: Mir, 1967. Pp. 316—342. (Translated into Russian).

Kharlou F. Kh. Chislennyy metod chastits v yacheykakh dlya zadach gidrodinamiki // Vychislitelnye metody v gidrodinamike. M.: Mir, 1967. S. 316—342.

5. *Bondarenko Yu. A.* Conservative splitting of the equation of energy in cruciform-type difference schemes for Lagrangian gas dynamics // Journal of Computational Mathematics and Mathematical Physics. 1997. V. 37, No. 8. Pp. 1020—1023. (In Russian).
   Bondarenko Yu .A. Konservativnoe rasshcheplenie uravneniya energii v raznostnykh skhemakh tipa "krest" dlya lagranzhevoy gazodinamiki // Zhurnal vychisl. mat. i mat. fiz. 1997. T. 37, № 8. S. 1020—1023.
6. *Monaghan J. J.* Smoothed particle hydrodynamics // Annu. Rev. Astron. Astrophys, 1992. Vol. 30. P. 543—574.
7. *Herant M., Benz W.* Postexplosion hydrodynamics of SN 1987A // Astrophys. J., 1992. Vol. 387. P. 294—308.
8. *Brookshow L.* Smooth particle hydrodynamics in cylindrical coordinates // ANZIAM J., 2003. Vol. 44 (E). P. C114—C139.
9. *Ba ê ta-Neves A. P., Ferreira A.* Shaped charge simulation using SPH in cylindrical coordinates // Int. J. for Comp.-Aided Eng. and Software. 2015. Vol. 32, No 2. P. 370—386.
10. *Richtmyer R., Morton K.* Difference methods for boundary-value problems. Moscow: Mir, 1972. (Translated into Russian).
    Rikhtmayer R., Morton K. Raznostnye metody resheniya kraevykh zadach. M.: Mir, 1972.
11. *Sedov L. I.* Methods of similarity and dimensionality in mechanics. Moscow: Nauka, 1981. (In Russian).
    Sedov L. I. Metody podobiya i razmernosti v mekhanike. M.: Nauka, 1981.

Received 03.07.19.